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ON THE STABILITY OF ELASTIC–PLASTIC AND RIGID–PLASTIC PLATES OF ARBITRARY THICKNESS, AND FLAT BARS OF ARBITRARY WIDTH

H. RAMSEY

Department of Mechanical Engineering, The University of British Columbia, Vancouver, Canada

Abstract—Constitutive relations in a form suitable for use in stability problems are derived for an elastic–plastic material undergoing a perturbation from a state of uniaxial stress, based on Green and Naghdi's general theory of an elastic–plastic continuum [1]. The deformation is considered to be quasi-static and isothermal, the material is assumed to yield according to the von Mises criterion, and to flow plastically according to the Reuss equations. The elastic strain rates are determined from Hooke's law. These relations are then applied to study the influence of plate thickness on the instability of a rectangular plate under uniform uniaxial compression. The instability of the surface of a compressed semi-infinite elastic–plastic continuum is then considered as the limiting case of infinite plate thickness. It is found that plastic flow has little influence on surface instability which is attributed to elastic deformation. Finally, two cases are considered for a rigid–plastic material, plane strain and plane stress, in order to compare the present treatment with previous work.

1. INTRODUCTION

ONE PARTICULAR objective of the present investigation is to determine whether plastic flow can lead to local instability of the surface of a compressed elastic–plastic body. Such instability would result in an apparent reduction of surface hardness, and the phenomenon could be of some interest in orogenic processes. Elastic materials exhibit this instability, as shown by the work of Biot [2, 3], John [4], and Kerr [5], but at a critical stress which is on the order of the elastic constants of the material. Rubber-like materials can sustain stresses on the order of the elastic constants, and surface instability is readily observable in these materials. Crystalline materials, however, cannot support such stresses, and so surface instability cannot occur physically as the result of elastic deformation. For ductile elastic–plastic materials, stresses can be on the order of the tangent modulus, and the question then arises whether plastic flow can lead to surface instability.

The instability of the surface of a compressed semi-infinite elastic–plastic continuum is taken as the limiting case of plate buckling as the ratio of plate thickness to buckling wavelength approaches infinity. In the following, two modes of plate buckling are treated in parallel, the antisymmetrical bending mode shown in Fig. 1, which is the usual buckling mode considered in the strength of materials treatment of plate buckling, and the symmetric bulging mode shown in Fig. 2. By basing the investigation on a plate of arbitrary thickness, it is possible to check the present treatment with the Euler buckling load for thin elastic plates, and the tangent modulus buckling load for thin elastic–plastic plates, in addition to providing results which apply to thick plates. The symmetric bulging mode may contribute to the bulging of specimens when tested in compression, as suggested by Goodier [6]. This bulging is usually attributed entirely to the restraint caused by friction between the specimen and the platens of the testing machine. When the direction of stress is reversed, the symmetric mode describes necking.

Agreement is found with the solution given by Cowper and Onat [7] for the instability of a rigid–plastic plate under plane strain. An extension of the results of [7] is suggested for the case where the ratio of plate thickness to buckling wavelength is greater than one-half. In the present treatment, the stress rates satisfy the condition of objectivity. For the case of plane stress, an interesting difference between [6] and the present treatment arises due to different stress rates being used in the von Mises yield condition, both of which are objective.

2. CONSTITUTIVE EQUATIONS

Following Green and Naghdi [1], we designate the current position of a material point of the continuum in a state B by the coordinates y_m referred to fixed rectangular cartesian axes, and the initial position of the same material point by the coordinates Y_M. As the deformation of the continuum proceeds, the coordinates Y_M define a convected curvilinear coordinate system related to fixed rectangular cartesian axes by

$$y_m = y_m(Y_1, Y_2, Y_3, t) \tag{2.1}$$

where t is time. The strain tensor e_{KL} is given by ([1], equation (2.17))

$$2e_{KL} = y_{k,K}y_{k,L} - \delta_{KL}. \tag{2.2}$$

In (2.2), $(\ )_{,M}$ denotes partial differentiation with respect to Y_M, the summation convention applies to the repeated index, and δ_{KL} is the Kronecker delta. The total strain is the sum of an elastic part denoted by a single prime, and a plastic part denoted by a double prime, ([1], equation (5.2)),

$$e_{KL} = e'_{KL} + e''_{KL} \tag{2.3}$$

We designate the covariant metric tensor of state B referred to the coordinate system Y_M by g_{KL}. Then

$$g_{KL} = \delta_{KL} + 2(e'_{KL} + e''_{KL}), \tag{2.4}$$

and since the elastic strains e'_{KL} are small, at least in crystalline materials,

$$g_{KL} \simeq \delta_{KL} + 2e''_{KL}. \tag{2.5}$$

The constitutive equation for the elastic strains is taken as the relation

$$Ee'_{KL} = [(1+\nu)g_{KM}g_{LN} - \nu g_{KL}g_{MN}]T^{MN} \tag{2.6}$$

where E is Young's modulus, ν is Poisson's ratio, and T^{MN} is the symmetric contravariant true stress tensor referred to the coordinate system Y_M. Equation (2.6) is similar in form to the usual statement of Hooke's law ([8], equation (5.4.32)), except that the metric tensor g_{KL} refers to the current deformed configuration of the continuum, rather than the initial state. For finite elastic–plastic deformations, the use of g_{KL} rather than δ_{KL} is appropriate. Since g_{KL} depends on e''_{KL}, from (2.5), the constitutive equation (2.6) takes account of the result established in [1] that during unloading the strain tensor depends on the existing plastic strain and is not just a function of stress, the temperature remaining constant. When approximations of the type (2.5) are allowed for, the constitutive relation (2.6) can be based on the Helmholtz function of Section 6 in [1] (see Appendix).

From (2.6), the constitutive equation for the elastic strain rates is

$$\begin{aligned} E\dot{e}'_{KL} = {} & [(1+\nu)(\dot{g}_{KM}g_{LN}+g_{KM}\dot{g}_{LN})-\nu(\dot{g}_{KL}g_{MN}+g_{KL}\dot{g}_{MN})]T^{MN} \\ & +[(1+\nu)g_{KM}g_{LN}-\nu g_{KL}g_{MN}]\dot{T}^{MN}. \end{aligned} \tag{2.7}$$

where the superposed dot designates partial differentiation with respect to t holding Y_M constant. An important feature of (2.7) is that the elastic strain rates depend on the plastic strain rates. There is a simple kinematic explanation for this. An elastic–perfectly plastic material, for example, flows under constant stress. The state of elastic strain remains constant, but the components of elastic strain e'_{KL} change since the convected coordinate system to which they refer is changing.

We proceed now to derive a constitutive relation for plastic strain rates which satisfies the condition of incompressibility and which is based on a yield condition that is independent of hydrostatic stress. We designate the stress tensor referred to the fixed rectangular cartesian axes y_m by σ_{ij} and define a plastic deformation rate d''_{ij} by ([1], equation (8.2))

$$\dot{e}''_{KL} = y_{i,K}y_{j,L}d''_{ij}. \tag{2.8}$$

The isothermal isotropic yield condition of von Mises given by

$$f = \tfrac{1}{2}(\delta_{im}\delta_{jn}-\tfrac{1}{3}\delta_{ij}\delta_{mn})\sigma_{ij}\sigma_{mn} \tag{2.9}$$

is independent of hydrostatic stress. We relate the yield condition (2.9) to d''_{ij} by the constitutive relation ([1], equation (8.25) and (8.26))

$$d''_{ij} = \lambda\beta_{ij}\frac{\partial f}{\partial\sigma_{mn}}\hat{\sigma}_{mn}, \qquad \left(\frac{\partial f}{\partial\sigma_{mn}}\hat{\sigma}_{mn} > 0\right) \tag{2.10}$$

where ([1], equation (8.23))

$$\hat{\sigma}_{ij} = \dot{\sigma}_{ij}+W_{mi}\sigma_{mj}+W_{mj}\sigma_{mi}. \tag{2.11}$$

In (2.11), W_{ij} is a skew-symmetric tensor depending on the motion of the continuum. We formulate the Reuss flow rule, that the plastic strain rates are proportional to the components of the deviatoric stress tensor, by putting

$$\beta_{ij} = (\delta_{im}\delta_{jn}-\tfrac{1}{3}\delta_{ij}\delta_{mn})\sigma_{mn} \tag{2.12}$$

in (2.10). Now the condition of incompressibility $d''_{ii} = 0$ ([1], equation (11.10)) is satisfied since $\beta_{ii} = 0$.

The current state B of the continuum is now restricted to be a state of uniform finite strain described by specializing (2.1) to the form

$$y_m = a_{mM}(t)Y_M \tag{2.13}$$

where

$$a_{mM}(t) = \begin{bmatrix} \lambda_1(t) & 0 & 0 \\ 0 & \lambda_2(t) & 0 \\ 0 & 0 & \lambda_3(t) \end{bmatrix},$$

$\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$ being the extension ratios. Since the material is isotropic, the stress tensor σ_{ij} has the corresponding form

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \tag{2.14}$$

For this special form of σ_{ij}, (2.14), it follows from (2.11) that

$$\hat{\sigma}_{11} = \dot{\sigma}_{11}, \qquad \hat{\sigma}_{22} = \dot{\sigma}_{22}, \qquad \hat{\sigma}_{33} = \dot{\sigma}_{33}. \tag{2.15}$$

In order to use the plastic deformation rates d''_{ij} in conjunction with the elastic strain rates $\dot{e}'_{KL}$ in (2.7), it is convenient to introduce a second system of convected coordinates θ_m. The covariant metric tensor referred to the coordinate system θ_m is denoted by G_{ij}, the covariant strain tensor by γ_{ij}, and the contravariant true stress tensor by τ^{ij}. We assume the deformation is uniform as described by (2.13) until $t = t_0$. At time t_0 the continuum is tested for instability by considering a bifurcation of (2.13). The convected system of coordinates θ_m at $t = t_0$ is related to the fixed rectangular cartesian axes y_m by

$$y_m = \theta_m, \qquad (t = t_0). \tag{2.16}$$

Hence

$$G_{ij} = \delta_{ij}, \qquad (t = t_0). \tag{2.17}$$

The plastic strain rate $\dot{\gamma}''_{ij}$ is related to $\dot{e}''_{KL}$ by

$$\dot{e}''_{KL} = \frac{\partial \theta^i}{\partial Y_K} \frac{\partial \theta^j}{\partial Y_L} \dot{\gamma}''_{ij} \tag{2.18}$$

where $\partial \theta^m / \partial Y_M$ is independent of t. Then from (2.8) and (2.16), it follows that

$$\dot{\gamma}''_{ij} = d''_{ij}, \qquad (t = t_0). \tag{2.19}$$

We denote the velocity of a material point referred to the fixed rectangular cartesian axes y_m by $\dot{y}_m$, and put

$$\dot{y}_m = w_m(\theta_1, \theta_2, \theta_3), \qquad (t = t_0). \tag{2.20}$$

The relation between the stress rates $\dot{\sigma}_{ij}$ and $\dot{\tau}^{ij}$ is obtained by differentiating the transformation

$$\sigma_{ij} = \frac{\partial y_i}{\partial \theta^m} \frac{\partial y_j}{\partial \theta^n} \tau^{mn} \tag{2.21}$$

partially with respect to t holding θ_m, and hence Y_M, constant. From (2.16) and (2.20), it follows that

$$\dot{\sigma}_{ij} = \frac{\partial w_i}{\partial \theta^m} \tau^{jm} + \frac{\partial w_j}{\partial \theta^m} \tau^{im} + \dot{\tau}^{ij}, \qquad (t = t_0). \tag{2.22}$$

We note also that

$$\sigma_{ij} = \tau^{ij}, \qquad (t = t_0). \tag{2.23}$$

The total strain rate $\dot{\gamma}_{ij}$ is simply related to the velocity field (2.20). From (2.3), (2.4) and equation (4.2.8) of [8], we find

$$\dot{G}_{ij} = 2\dot{\gamma}_{ij} = \frac{\partial w_i}{\partial \theta^j} + \frac{\partial w_j}{\partial \theta^i}, \qquad (t = t_0). \tag{2.24}$$

3. PLANE STRAIN PERTURBATION SUPERPOSED ON SUSTAINED FLOW UNDER UNIAXIAL STRESS

A rectangular plate of length $2L$, thickness $2h$, and width $2b$ much greater than the thickness at time t_0 is under uniform compressive stress P acting parallel to the length. The fixed rectangular cartesian axes y_m are oriented so that the y_1 axis is in the direction of the length of the plate, and the y_2 axis is in the direction of the thickness, Figs. 1 and 2. From (2.23), it follows that $\tau^{11} = -P$, and the other components of τ^{ij} are zero. The compressed edges of the plate are the material surfaces $\theta_1 = \pm L$, and the material surfaces $\theta_2 = \pm h$ of the plate remain free from traction. A state of generalized plane strain subsequent to time t_0 is specified by writing the functions w_m in (2.20) as

$$w_1 = u(x, y) \qquad w_2 = v(x, y) \qquad w_3 = w(z) \tag{3.1}$$

and the stress rates $\dot{\tau}^{ij}$ as

$$\begin{aligned} &\dot{\tau}^{11} = \dot{\tau}^{11}(x, y) \qquad \dot{\tau}^{22} = \dot{\tau}^{22}(x, y) \qquad \dot{\tau}^{33} = \dot{\tau}^{33}(x, y) \\ &\dot{\tau}^{12} = \dot{\tau}^{12}(x, y) \qquad \dot{\tau}^{23} = \dot{\tau}^{31} = 0. \end{aligned} \tag{3.2}$$

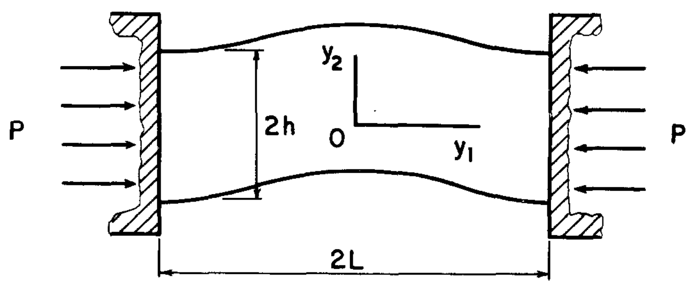

FIG. 1. Antisymmetric mode.

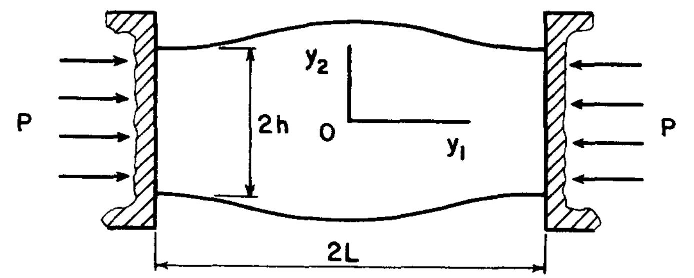

FIG. 2. Symmetric mode.

In (3.1) and (3.2), we have put $\theta_1 = x$, $\theta_2 = y$, $\theta_3 = z$. The elastic strain rates $\dot{\gamma}'_{ij}$, in the presence of the prestress $\tau^{11} = -P$ are, from (2.7) and (2.17),

$$\begin{aligned} E\dot{\gamma}'_{11} &= -2P\dot{G}_{11} + \dot{\tau}^{11} - \nu(\dot{\tau}^{22} + \dot{\tau}^{33}) \\ E\dot{\gamma}'_{22} &= \nu P(\dot{G}_{11} + \dot{G}_{22}) + \dot{\tau}^{22} - \nu(\dot{\tau}^{33} + \dot{\tau}^{11}) \\ E\dot{\gamma}'_{33} &= \nu P(\dot{G}_{11} + \dot{G}_{33}) + \dot{\tau}^{33} - \nu(\dot{\tau}^{11} + \dot{\tau}^{22}) \\ E\dot{\gamma}'_{12} &= -P\dot{G}_{12} + (1+\nu)\dot{\tau}^{12}, \qquad (t = t_0) \end{aligned} \tag{3.3}$$

and the plastic strain rates $\dot{\gamma}''_{ij}$, from (2.9), (2.10), (2.12), (2.15), (2.19), and (2.22), are

$$\begin{aligned} \dot{\gamma}''_{11} &= -\frac{1}{2H}\left[4P\frac{\partial u}{\partial x} - 2\dot{\tau}^{11} + \dot{\tau}^{22} + \dot{\tau}^{33}\right] \\ \dot{\gamma}''_{22} = \dot{\gamma}''_{33} &= \frac{1}{4H}\left[4P\frac{\partial u}{\partial x} - 2\dot{\tau}^{11} + \dot{\tau}^{22} + \dot{\tau}^{33}\right] \\ \dot{\gamma}''_{12} &= 0, \qquad (t = t_0). \end{aligned} \tag{3.4}$$

where $H = 9/(4\lambda P^2)$. Combining (3.3) and (3.4), using (2.3), (2.24), and (3.1), and collecting terms, we obtain:

$$\begin{aligned} &\left(\frac{1}{H} + \frac{1}{E}\right)\dot{\tau}^{11} - \left(\frac{1}{2H} + \frac{\nu}{E}\right)\dot{\tau}^{22} - \left(\frac{1}{2H} + \frac{\nu}{E}\right)\dot{\tau}^{33} = \left(1 + \frac{4P}{E} + \frac{2P}{H}\right)\dot{\gamma}_{11} \\ -&\left(\frac{1}{2H} + \frac{\nu}{E}\right)\dot{\tau}^{11} + \left(\frac{1}{4H} + \frac{1}{E}\right)\dot{\tau}^{22} + \left(\frac{1}{4H} - \frac{\nu}{E}\right)\dot{\tau}^{33} = \left(1 - \frac{2\nu P}{E}\right)\dot{\gamma}_{22} - \left(\frac{2\nu P}{E} + \frac{P}{H}\right)\dot{\gamma}_{11} \\ -&\left(\frac{1}{2H} + \frac{\nu}{E}\right)\dot{\tau}^{11} + \left(\frac{1}{4H} - \frac{\nu}{E}\right)\dot{\tau}^{22} + \left(\frac{1}{4H} + \frac{1}{E}\right)\dot{\tau}^{33} = \left(1 - \frac{2\nu P}{E}\right)\dot{\gamma}_{33} - \left(\frac{2\nu P}{E} + \frac{P}{H}\right)\dot{\gamma}_{11} \end{aligned} \tag{3.5}$$

$$\dot{\tau}^{12} = \frac{E}{1+\nu}\left(1 + \frac{2P}{E}\right)\dot{\gamma}_{12}. \tag{3.6}$$

The quantity H in (3.4) and (3.5), which in turn is related to λ, can now be referred to the slope of the uniaxial stress–strain curve. For continuing uniaxial stress, the first of (3.5) reduces to

$$\left(\frac{1}{H} + \frac{1}{E}\right)\dot{\tau}^{11} = \left(1 - \frac{4\sigma}{E} - \frac{2\sigma}{H}\right)\dot{\gamma}_{11} \tag{3.7}$$

where we have put $-P = \sigma$. The tangent modulus E_T is defined as the slope of the physical stress (force/current area) versus logarithmic strain curve. The physical stress σ and the stress rate $\dot{\sigma}$ coincide with σ_{11} and $\dot{\sigma}_{11}$ respectively. Then

$$\dot{\sigma} = E_T\dot{\gamma}_{11} \tag{3.8}$$

since $\dot{\gamma}_{11} = \partial u/\partial x$, the rate of increase in length per current unit length which is the logarithmic strain rate. From (2.22), (3.7), and (3.8) it follows that

$$\frac{1}{H} = \frac{1}{E_T} - \frac{1}{E} - \frac{2\sigma}{EE_T}. \tag{3.9}$$

The plastic behaviour of the material is now described by a single parameter, the tangent modulus E_T, which is assumed to be known as a function of stress σ (tension) or P (compression). For a rigid–plastic material ($E \to \infty$), H coincides with the tangent modulus E_T.

The equations of equilibrium for the stress rates $\dot{t}^{ij}$, for zero body force, are ([8], equation (4.2.21))

$$\left.\begin{aligned} -2P\frac{\partial^2 u}{\partial x^2}-P\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial \dot{t}^{11}}{\partial x}+\frac{\partial \dot{t}^{12}}{\partial y} &= 0 \\ -P\frac{\partial^2 v}{\partial x^2}+\frac{\partial \dot{t}^{12}}{\partial x}+\frac{\partial \dot{t}^{22}}{\partial y} &= 0. \end{aligned}\right\}(t = t_0) \tag{3.10}$$

These equations depend upon (2.16) and (2.20), and so apply to a small perturbation from the uniform state B which is assumed to be an equilibrium state. The convected surfaces $\theta_2 = y = \pm h$ of the plate remain free from traction provided ([8], equation (4.2.30))

$$\dot{t}^{12} = \dot{t}^{22} = 0 \qquad (t = t_0) \tag{3.11}$$

at $y = \pm h$.

The constitutive relations (3.5) and (3.6), the differential equations of equilibrium (3.10), and the boundary conditions (3.11) are linear in the stress rates $\dot{t}^{ij}$ and the strain rates $\dot{\gamma}_{ij}$. Hence the deformation may be treated as plane strain deformation described by the velocities u, v superposed on uniform deformation under increasing uniaxial stress. Accordingly we put $\dot{\gamma}_{33} = 0$ in (3.5), and eliminate $\dot{t}^{33}$ since it does not appear in (3.10) or (3.11). Hence (3.5) reduce to

$$\begin{aligned} \dot{t}^{11}+2\dot{t}^{22} &= \frac{E}{1-2\nu}\left(1+\frac{4P}{E}-\frac{4\nu P}{E}\right)\dot{\gamma}_{11}+\frac{E(2-\nu)}{(1+\nu)(1-2\nu)}\left(1-\frac{2\nu P}{E}\right)\dot{\gamma}_{22} \\ -\left(\frac{1}{H}+\frac{1}{E}\right)\dot{t}^{11}+\left(\frac{1}{H}+\frac{2\nu}{E}\right)\dot{t}^{22} &= -\left(1+\frac{2P}{H}+\frac{4P}{E}\right)\dot{\gamma}_{11} \\ &\quad +\frac{1}{1+\nu}\left(\nu+\frac{E}{2H}\right)\left(1-\frac{2\nu P}{E}\right)\dot{\gamma}_{22}. \end{aligned} \tag{3.12}$$

In (3.5). (3.6) and (3.12) the strain rates $\dot{\gamma}_{ij}$ are related to the velocities (2.20) by (2.24).

4. SOLUTION OF THE STABILITY PROBLEM

We now seek a solution of (2.24), (3.6), (3.10) and (3.12) satisfying the boundary conditions (3.11) of the form

$$\begin{aligned} u &= A\,e^{ry}\sin\alpha x \\ v &= B\,e^{ry}\cos\alpha x \\ \dot{t}^{11} &= C\,e^{ry}\cos\alpha x \\ \dot{t}^{22} &= D\,e^{ry}\cos\alpha x \end{aligned} \tag{4.1}$$

where A, B, C, D, and r are constants, and $\alpha = \pi/L$. The expression for u has the property that $u = 0$ when $x = \pm L$. Thus if the plate is being compressed by rigid frictionless dies along the edges $x = \pm L$, (4.1) describes a bifurcation from a state of uniform uniaxial stress. For non-zero A, B, C, D, from (2.24), (3.6), (3.10) and (3.12) we must have

$$\left(1+\frac{E}{4H}\cdot\frac{5-4\nu}{1-\nu^2}\right)\left(\frac{r}{\alpha}\right)^4-2\left(1-\frac{E}{4H}\cdot\frac{1}{1-\nu}\right)\left(\frac{r}{\alpha}\right)^2+\left(1+\frac{E}{2H}\cdot\frac{1}{1-\nu}\right)=0. \tag{4.2}$$

The four roots of (4.2) may be written

$$\frac{r}{\alpha} = \pm(a \pm ib), \qquad (i = \sqrt{-1}) \tag{4.3}$$

where a, b are real positive numbers given by

$$\begin{aligned} a^2-b^2 &= \frac{1-[E/4H(1-\nu)]}{1+[E(5-4\nu)/4H(1-\nu^2)]} \\ (a^2+b^2)^2 &= \frac{1+[E/2H(1-\nu)]}{1+[E(5-4\nu)/4H(1-\nu^2)]} \end{aligned} \tag{4.4}$$

(i) *Elastic case*

The plastic strain rates can be set equal to zero by putting $1/H = 0$. The roots of (4.2) for an elastic material are coincident with $r/\alpha = \pm 1$. The velocities u, v then may take the form

$$\begin{aligned} u &= (A_1 \sinh \alpha y + A_2 \alpha y \cosh \alpha y) \sin \alpha x \\ v &= (B_1 \cosh \alpha y + B_2 \alpha y \sinh \alpha y) \cos \alpha x \end{aligned} \tag{4.5}$$

or

$$\begin{aligned} u &= (A_1 \cosh \alpha y + A_2 \alpha y \sinh \alpha y) \sin \alpha x \\ v &= (B_1 \sinh \alpha y + B_2 \alpha y \cosh \alpha y) \cos \alpha x \end{aligned} \tag{4.6}$$

where

$$B_1 = \frac{1+(2P/E)}{1-(2\nu P/E)}[-A_1+(3-4\nu)A_2]$$

$$-B_2 = \frac{1+(2P/E)}{1-(2\nu P/E)}A_2.$$

In (4.5), v is an even function of y and hence describes the antisymmetrical "buckling" mode shown in Fig. 1; in (4.6), v is an odd function of y and describes the symmetric "bulging" mode shown in Fig. 2. The boundary conditions (3.11) can be satisfied for non-zero A_1, A_2, provided

$$\pm\frac{\sinh(2\alpha h)}{2\alpha h}-1 = \frac{(1-\nu^2)(2P/E)}{1-(1-\nu)\cdot(1+2\nu)(P/E)}. \tag{4.7}$$

The positive sign on the left hand side of (4.7) should be taken for the antisymmetric mode described by (4.5), and the negative sign for the symmetric mode described by (4.6).

We now take the positive sign on the left hand side of (4.7), and retain only the leading term of the power series expansion of the left hand side in αh; also we take only the leading term of the power series expansion of the right hand side in P/E. Thus we obtain the Euler formula for a thin plate of thickness $2h$ and length $2L$ buckling into a cylindrical form, the compressed edges $x = \pm L$ being built-in against rotation,

$$\frac{2\pi^2 h^2}{3L^2} = (1-\nu^2)\frac{2P}{E}, \qquad \left(L = \frac{\pi}{\alpha}\right). \tag{4.8}$$

Since the power series expansion of the left hand side of (4.7) is composed entirely of positive terms, the solution (4.7) yields a higher buckling stress than the strength of materials solution (4.8) which neglects shear deformation.

The vanishing of the denominator of the right hand side of (4.7) corresponds to the local instability of the surface of a compressed semi-infinite elastic continuum ($h/L \to \infty$). The critical stress P is approximately equal to the Young's modulus E,

$$\frac{P}{E} = \frac{1}{(1-\nu)(1+2\nu)}. \tag{4.9}$$

Such a stress is unrealistic for crystalline materials but is possible in rubber-like materials. The surface instability of rubber in compression is discussed by Biot [3], who uses an appropriate strain energy function rather than Hooke's law.

Solutions of (4.7) for the symmetric mode exist for

$$\frac{P}{E} > \frac{1}{(1-\nu)\cdot(1+2\nu)} \tag{4.10}$$

the value of P/E increasing as h/L decreases. There is no solution for this mode for $P/E < 0$ (necking in tension).

(ii) *Elastic–plastic case*

It is convenient now to express the velocities u, v in the form

$$\begin{aligned} u &= (A \sinh \rho\alpha y + A^* \sinh \rho^*\alpha y) \sin \alpha x \\ v &= (B \cosh \rho\alpha y + B^* \cosh \rho^*\alpha y) \cos \alpha x \end{aligned} \tag{4.11}$$

for the antisymmetric mode, Fig. 1, and

$$\begin{aligned} u &= (A \cosh \rho\alpha y + A^* \cosh \rho^*\alpha y) \sin \alpha x \\ v &= (B \sinh \rho\alpha y + B^* \sinh \rho^*\alpha y) \cos \alpha x \end{aligned} \tag{4.12}$$

for the symmetric mode, Fig. 2. In (4.11) and (4.12)

$$\rho = a + ib, \tag{4.13}$$

and A, B are complex constants; ρ^*, A^*, B^*, are the complex conjugates of ρ, A, B. The stress rates $\dot{t}^{11}$, $\dot{t}^{12}$, $\dot{t}^{22}$, may be expressed in terms of A, A^*, B, B^*, using (2.24), (3.6) and

(3.10). We then put $y = \pm h$ and consider the set of four equations (3.11) and (3.12) which are homogeneous in A, A^*, B, B^*. For non-zero values of these constants,

$$\pm\frac{b\sinh(2a\alpha h)}{a\sin(2b\alpha h)}-1 = \frac{U}{V}$$

where

$$\begin{aligned} U &= (a^2+b^2)[(1-\nu^2)+(E/4H)(5-4\nu)](2P/E) \\ V &= 1-(1-\nu)(1+2\nu)(P/E)+(1-\nu^2)(P/E)[1-(a^2+b^2)] \\ &\quad +(E/4H)[1+(2P/E)-(5-4\nu)(a^2+b^2)(P/E)] \end{aligned} \tag{4.14}$$

Again, the plus sign on the left hand side of (4.14) applies for the antisymmetric mode, and the minus sign for the symmetric mode. As a check, the eigencondition for the elastic case (4.7) may be recovered from (4.14) by putting $1/H = 0$, $a = 1$, and taking the limit as $b \to 0$.

For the buckling of a thin plate, retaining the leading term in the power series expansion of each side of (4.14) yields, after cancelling the factor (a^2+b^2) which is common to both sides,

$$\frac{2\pi^2h^2}{3L^2} = \frac{[(1-\nu^2)+(E/4H)(5-4\nu)](2P/E)}{1+(E/4H)} \tag{4.15}$$

The result (4.15) is the "tangent modulus" load which may be defined in analogy to the tangent modulus load for columns. When a column buckles without reversal in the direction of straining, the incremental stress–strain relation for the superposed bending deformation is governed by the tangent to the stress–strain curve for uniaxial stress. For a rectangular plate buckling into a cylindrical form, the superposed bending deformation occurs under the constraint of plane strain. Putting $\dot{\tau}^{22} = 0$ in (3.12), eliminating $\dot{\gamma}_{22}$, and neglecting terms in P, we find

$$\dot{\tau}^{11} = \frac{E[1+(E/4H)]}{(1-\nu^2)+(E/4H)(5-4\nu)}\dot{\gamma}_{11}. \tag{4.16}$$

The coefficient of $\dot{\gamma}_{11}$ in (4.16) is the tangent modulus for plane strain. When $E/(1-\nu^2)$ in (4.8) is replaced by the coefficient of $\dot{\gamma}_{11}$ in (4.16), (4.15) is again obtained.

For ductile metals under uniaxial stress, the maximum value of P/E that may be expected without some form of rupture occurring may be assumed to be about ± 5 per cent. Hence the right hand side of (4.15) multiplied by (a^2+b^2) always reasonably approximates the right hand side of (4.14). At the outset of yielding, $a = 1$ and $b = 0$. As the tangent modulus E_T decreases, a diminishes continuously and b increases continuously. As E_T becomes small, a and b approach limiting values governed by Poisson's ratio ν, (3.9) and (4.4). For $\nu = \frac{1}{4}$, these limiting values are approximately $a = \frac{1}{2}$ and $b = \frac{3}{4}$. The left hand side of (4.14) becomes infinite as h/L increases to $\frac{2}{3}$, while the right hand side is of the order of P/E. Hence solutions of (4.14) for buckling exist only for $h/L \ll \frac{2}{3}$. There are no solutions of (4.14) within the reasonable range of values of P/E for the symmetric mode, for P/E either positive or negative. The denominator of the right hand side of (4.14) has a zero which is essentially independent of the tangent modulus E_T and which almost coincides with the value of P/E given by (4.9). Solutions of (4.14) for the symmetric mode exist for values of P/E greater than the value for which the denominator on the right hand side of (4.14) vanishes, as for the elastic case. However, such large stresses are physically unrealistic, and accordingly this mode of instability could not occur in practice.

(iii) *Rigid–plastic case*

The limit of the right hand side of (4.14) as Young's modulus E increases indefinitely is zero, and hence no solutions of (4.14) exist for the case of a plate of rigid–plastic material. This result is in agreement with the expressions given by Hill [9], which describe the most general velocity field for a rigid–plastic body under uniform uniaxial stress. These expressions contain just quadratic and linear terms in the current rectangular cartesian coordinates of a material point, and hence exclude the sinusoidal fields (4.11) and (4.12). However, there are two cognate problems for a rigid–plastic material for which solutions can be found for both the antisymmetric and symmetric modes, plane strain and plane stress.

The fact that the solution of (4.14) in the limit as h/L increases indefinitely is almost independent of the tangent modulus E_T and practically coincides with that given by (4.9) for the entirely elastic material, and also the negative result for the rigid–plastic material suggest that the surface instability indicated by the solution of (4.14) in limit arises from elastic deformation of the material. Hence the surface instability exhibited by rubber-like materials could not occur in crystalline materials, even when plastic flow is considered, since the critical compressive stress is physically unattainable, being on the order of the elastic constants.

5. INSTABILITY IN PLANE STRAIN OF A RIGID-PLASTIC MATERIAL

The case of plane strain has been considered by Cowper and Onat [7] who base their analysis on the von Mises yield condition, and refer stress and strain rates to fixed rectangular cartesian axes (Onat, [10]). Formulas are given for the critical stress to initiate necking, and also lateral buckling, which apply when $h/L < \frac{1}{2}$. In the following, a solution in terms of the present treatment is stated briefly. The eigenconditions obtained are exactly those given in [7], only an extension of the results to the range $h/L > \frac{1}{2}$ is suggested.

If state B at time t_0 is a state of uniform plane strain rather than uniform uniaxial stress, we take $\tau^{33} = -P/2$, in order that $\beta_{33} = 0$ in (2.12). As before $\tau^{11} = -P$. Hence instead of (3.4) we have

$$\dot{\gamma}''_{11} = -\dot{\gamma}''_{22} = -\frac{3}{4}\frac{1}{H}\left[2P\frac{\partial u}{\partial x} - \dot{\tau}^{11} + \dot{\tau}^{22}\right] \tag{5.1}$$

For convenience, we put

$$\mu = \tfrac{4}{3}H = \tfrac{4}{3}E_T, \tag{5.2}$$

μ being the "tangent modulus" for plane strain, corresponding to $E/(1-\nu^2)$ for an elastic material. Now $H = E_T$, since $1/E = 0$ in (3.9). We rewrite (5.1) as

$$-(2P+\mu)\frac{\partial u}{\partial x} + \dot{\tau}^{11} - \dot{\tau}^{22} = 0 \tag{5.3}$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{5.4}$$

Since all elastic strain rates are now zero, $\dot{\gamma}_{12} = \dot{\gamma}''_{12} = 0$, from (3.6), and hence

$$\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y} = 0. \tag{5.5}$$

We now seek a solution of the set of five equations (3.10), (5.3), (5.4) and (5.5) satisfying the boundary conditions $u = 0$ at $x = \pm L$ and (3.11). Since these equations and the boundary conditions are all linear in the five unknowns u, v, $\dot{t}^{11}$, $\dot{t}^{12}$, $\dot{t}^{22}$, the deformation they describe may be superposed on uniform deformation due to increasing P.

For the antisymmetric mode, u and v satisfying (5.4) and (5.5) may be written

$$\begin{aligned} u &= C_1 \sin \alpha y \sin \alpha x \\ v &= C_1 \cos \alpha y \cos \alpha x \end{aligned} \tag{5.6}$$

where C_1 is an arbitrary constant. The corresponding expressions for $\dot{t}^{11}$, $\dot{t}^{12}$, $\dot{t}^{22}$ satisfying (3.10) and (5.3) are

$$\begin{aligned} \dot{t}^{11} &= \{[(2P+\mu)\alpha C_1 + C_2] \sin \alpha y + \tfrac{1}{2}\mu\alpha^2 C_1 y \cos \alpha y\} \cos \alpha x \\ \dot{t}^{12} &= -\{[(P+\mu/2)\alpha C_1 + C_2] \cos \alpha y - \tfrac{1}{2}\mu\alpha^2 C_1 y \sin \alpha y\} \sin \alpha x \\ \dot{t}^{22} &= \{\tfrac{1}{2}\mu\alpha^2 C_1 y \cos \alpha y + C_2 \sin \alpha y\} \cos \alpha x, \end{aligned} \tag{5.7}$$

where C_2 is also an arbitrary constant. Similarly, for the symmetric mode, we have

$$\begin{aligned} u &= C_1 \cos \alpha y \sin \alpha x \\ v &= -C_1 \sin \alpha y \cos \alpha x \\ \dot{t}^{11} &= \{[(2P+\mu)\alpha C_1 + C_2] \cos \alpha y - \tfrac{1}{2}\mu\alpha^2 C_1 y \sin \alpha y\} \cos \alpha x \\ \dot{t}^{12} &= \{[(P+\mu/2)\alpha C_1 + C_2] \sin \alpha y + \tfrac{1}{2}\mu\alpha^2 C_1 y \cos \alpha y\} \sin \alpha x \\ \dot{t}^{22} &= \{-\tfrac{1}{2}\mu\alpha^2 C_1 y \sin \alpha y + C_2 \cos \alpha y\} \cos \alpha x. \end{aligned} \tag{5.8}$$

For C_1, $C_2 \neq 0$ in (5.7) and (5.8), satisfying the boundary conditions (3.11) requires

$$\pm\frac{\sin 2\alpha h}{2\alpha h} = \frac{1}{1+2P/\mu} \tag{5.9}$$

where the plus sign on the left hand side applies for the antisymmetric mode, and the minus sign for the symmetric mode. The eigenconditions (5.9) correspond exactly with those of [7] (equations (21) and (36)). In [7], the tangent modulus in pure shear h is related to the tangent modulus in uniaxial stress E_T by

$$h = \tfrac{2}{3}E_T \tag{5.10}$$

and the shear stress k is $P/2$. Hence

$$\frac{k}{h} = \frac{3}{4}\frac{P}{E_T}. \tag{5.11}$$

The left hand side of (5.9) is shown in Fig. 3, ($\phi = 2\alpha h = 2\pi h/L$), and the right hand side in Fig. 4 ($S = 2P/\mu = 3P/2E_T$). There are no solutions of (5.9) for

$$-2 < S < 0.$$

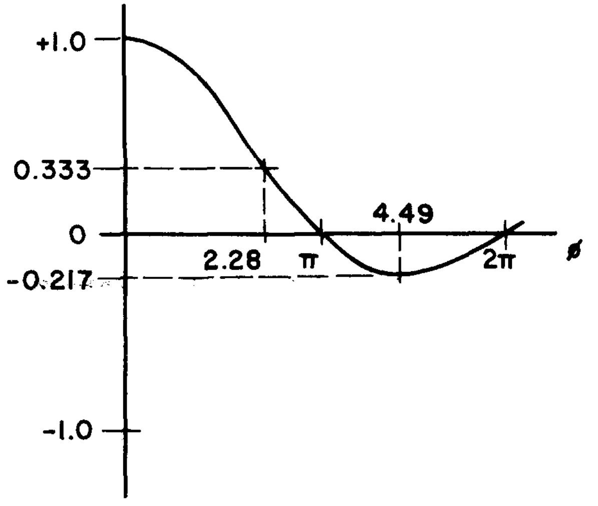

FIG. 3. The function sin ϕ/ϕ.

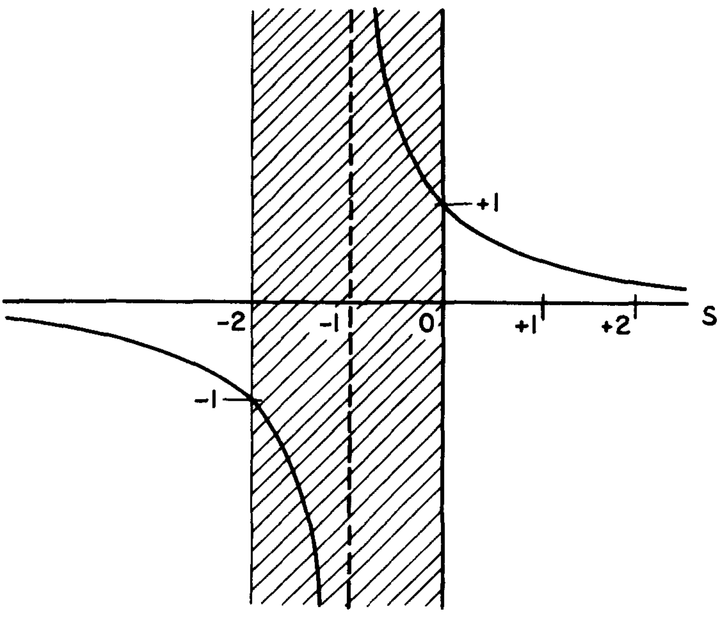

FIG. 4. The function $1/(1+S)$.

For S small and positive, the only solutions of (5.9) are for the antisymmetric mode. Retaining only the leading terms in the power series expansions in ϕ and S respectively for each side, we have, for buckling, $\phi^2/6 = S$, or

$$\frac{P}{E_T} = \frac{4\pi^2 h^2}{9L^2} \qquad \left(\frac{h}{L} \ll 1\right) \tag{5.12}$$

which gives the critical stress for the tangent modulus buckling load for plane strain. As ϕ increases, the critical value of S increases, S becoming infinite for $\phi = \pi$ ($h/L = \frac{1}{2}$). For

$$n < \frac{2h}{L} < (n+1) \qquad (n = 0, 2, 4 \ldots) \tag{5.13}$$

the plate is unstable in the antisymmetric buckling mode, and for

$$n < \frac{2h}{L} < (n+1) \qquad (n = 1, 3, 5 \ldots) \tag{5.14}$$

the plate is unstable in the symmetric bulging mode. The lowest value of S for the bulging mode occurs for $\phi = 4{\cdot}49$; then $S = 3{\cdot}60$. For "bulging" under the least compressive stress in plane strain,

$$\frac{h}{L} = 0{\cdot}714, \qquad \frac{P}{E_T} = 2{\cdot}40. \tag{5.15}$$

As ϕ and hence h/L tend to infinity, the critical compressive stress tends to infinity. A semi-infinite continuum of rigid–plastic material under compression does not exhibit surface instability as does an elastic continuum, at least if plastic flow is based on the von Mises yield criterion and associated flow rule. For a very thick plate of elastic or elastic–plastic material, the deformation due to instability is localized near the surface, whereas for a rigid–plastic plate, such deformation does not diminish with increasing depth, *cf.* (4.5), (4.11), and (5.6).

For S negative (tension), the symmetric mode which describes necking is the mode of instability for maximum S (minimum tension). Further, the tensile stress which causes necking increases as ϕ and hence h/L increase. For plane strain the minimum tensile stress σ ($\sigma = -P$) to produce necking is given by

$$\frac{\sigma}{\mu} = \frac{3\sigma}{4E_T} = 1, \qquad \left(\frac{h}{L} \ll 1\right). \tag{5.16}$$

For a plate under tension, (5.13) applies to the symmetric mode, and (5.14) to the antisymmetric mode.

Sewell [11] has disagreed with the plane strain solution of [7]. In general, the stress rates $\dot{\sigma}_{ij}$ referred to fixed rectangular cartesian axes are not objective; but since the principal axes of the stress tensor σ_{ij}, viz. (2.14), coincide with the fixed rectangular cartesian axes y_m, it follows from (2.11), since W_{ij} is skew-symmetric, that the objective stress rates $\hat{\sigma}_{11}$, $\hat{\sigma}_{22}$, $\hat{\sigma}_{33}$ coincide with the stress rates $\dot{\sigma}_{11}$, $\dot{\sigma}_{22}$, $\dot{\sigma}_{33}$ respectively. For the state of stress described by (2.14) and the von Mises yield condition described by (2.9), the stress rates $\hat{\sigma}_{11}, \hat{\sigma}_{22}, \hat{\sigma}_{33}$ are the only ones which enter into the constitutive relation (2.10). On the other hand, the shear stress rate $\hat{\sigma}_{12}$ as given by (2.11) does not coincide with $\dot{\sigma}_{12}$, but $\hat{\sigma}_{12}$ does not enter into (2.10) because $\partial f/\partial\sigma_{12} = 0$. Thus there is agreement between [7] and the treatment herein.

6. INSTABILITY IN PLANE STRESS OF A RIGID–PLASTIC MATERIAL

We consider now a flat bar of rigid–plastic material under uniform uniaxial stress in state B. The orientation of the fixed rectangular cartesian axes y_m is chosen as in Section 3, only now the dimension $2h$ parallel to the y_2 axis is referred to as the width, and the dimension $2b$ parallel to the y_3 axis is the thickness. All the components of the stress tensor τ^{ij} of state B are zero except τ^{11}. The thickness is restricted to be small in comparison with the length of the bar ($b/L \ll 1$), but the width $2h$ is allowed to vary arbitrarily, $0 < h < \infty$. Variations in stress and strain are anticipated to occur predominately in the y_1–y_2 plane, with negligible variation through the thickness parallel to the y_3 axis.

Since the state of stress in state B is uniaxial, it remains to justify the anticipation of sinusoidal velocity fields which describe buckling and necking, in view of Hill's demonstration in [9] that the most general exact fields are "quadratic". It follows from (2.12) that all the shear strain rates are zero, giving

$$\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y} = 0 \tag{6.1}$$

and

$$\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z} = 0, \qquad \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x} = 0 \tag{6.2}$$

which follow from (2.24). Since $\partial v/\partial y = \partial w/\partial z$, the condition of incompressibility becomes

$$\frac{\partial u}{\partial x}+2\frac{\partial v}{\partial y} = 0. \tag{6.3}$$

A solution of (6.1) and (6.3) may be written

$$\begin{aligned} u &= \sqrt{2}C_1 \sin\frac{\alpha y}{\sqrt{2}} \sin \alpha x, \qquad \left(\alpha = \frac{\pi}{L}\right) \\ v &= C_1 \cos\frac{\alpha y}{\sqrt{2}} \cos \alpha x. \end{aligned} \tag{6.4}$$

Then,

$$\frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = -\frac{\alpha}{\sqrt{2}}C_1 \sin\frac{\alpha y}{\sqrt{2}} \cos \alpha x \tag{6.5}$$

and, integrating,

$$w = -\left(\frac{\alpha}{\sqrt{2}}C_1 \sin\frac{\alpha y}{\sqrt{2}} \cos \alpha x\right) . z+f(x, y) \tag{6.6}$$

where $f(x, y)$ is some arbitrary function. Equations (6.2) become, after substitution from (6.4) and (6.6),

$$\begin{aligned} -\left(\frac{\alpha\pi}{2}C_1 \cos\frac{\alpha y}{\sqrt{2}} \cos \alpha x\right)\frac{z}{L}+\frac{\partial f}{\partial y} &= 0 \\ \left(\frac{\alpha\pi}{\sqrt{2}}C_1 \sin\frac{\alpha y}{\sqrt{2}} \sin \alpha x\right)\frac{z}{L}+\frac{\partial f}{\partial x} &= 0. \end{aligned} \tag{6.7}$$

Since $z/L \leq b/L$, the function $f(x, y)$ reduces to a constant in the limit as $b/L \to 0$, and so may be taken as zero for a sufficiently thin plate, $b/L \ll 1$. Thus the plane stress approximation considered here bears a close similarity to the plane stress approximation in infinitesimal elasticity. Equations (6.1) and (6.2) are hyperbolic partial differential equations for u and v, with characteristics $y \pm \sqrt{2}x$ = constant. These characteristics appear as slip-lines inclined at 55° to the y_1 axis, as observed in tests (Nadai [12], Figs. 19–28).

The edges of the bar, $\theta_2 = y = \pm h$, and the lateral surfaces, $\theta_3 = z = \pm b$, are to remain free from surface tractions. These conditions are satisfied provided ([8], equation (4.2.30)),

$$\dot{\tau}^{12} = \dot{\tau}^{22} = \dot{\tau}^{32} = 0 \quad \text{at } y = \pm h \tag{6.8}$$

and

$$\dot{\tau}^{13} = \dot{\tau}^{23} = \dot{\tau}^{33} = 0 \quad \text{at } z = \pm b. \tag{6.9}$$

A solution of the equations of equilibrium and the constitutive relations satisfying the boundary conditions (6.8) and (6.9) can be found by taking the stress rates $\dot{\tau}^{13}$, $\dot{\tau}^{23}$, $\dot{\tau}^{33}$ identically zero. The equations of equilibrium corresponding to (3.10) are then ([8], equation (4.2.21))

$$\begin{aligned} -P\frac{\partial^2 u}{\partial x^2} + \frac{\partial \dot{\tau}^{11}}{\partial x} + \frac{\partial \dot{\tau}^{12}}{\partial y} &= 0 \\ -P\frac{\partial^2 v}{\partial x^2} + \frac{\partial \dot{\tau}^{12}}{\partial x} + \frac{\partial \dot{\tau}^{22}}{\partial y} &= 0 \end{aligned} \tag{6.10}$$

where $-P = \tau^{11}$. The first of (3.4) becomes, putting $\dot{\tau}^{33} = 0$,

$$-(E_T + 2P)\frac{\partial u}{\partial x} + \dot{\tau}^{11} - \tfrac{1}{2}\dot{\tau}^{22} = 0. \tag{6.11}$$

Equations (6.1), (6.3), (6.10), and (6.11) determine the five unknown u, v, $\dot{\tau}^{11}$, $\dot{\tau}^{12}$, $\dot{\tau}^{22}$. The stress rates satisfy the boundary conditions (6.8), and $u = 0$ at $x = \pm L$. The velocities (6.4) describe the antisymmetric "bucking" mode, Fig. 1. The corresponding stress rates are, from (6.10) and (6.11),

$$\begin{aligned} \dot{\tau}^{11} &= \left\{[(2P + E_T)\sqrt{2}\alpha C_1 + \tfrac{1}{2}C_2] \sin\frac{\alpha y}{\sqrt{2}} + \left(\frac{P}{2} + E_T\right)\frac{\alpha}{\sqrt{2}} C_1 \frac{\alpha y}{\sqrt{2}} \cos\frac{\alpha y}{\sqrt{2}}\right\} \cos\alpha x \\ \dot{\tau}^{12} &= -\left\{\left[\left(\frac{3P}{2} + E_T\right)\alpha C_1 + \frac{1}{\sqrt{2}}C_2\right] \cos\frac{\alpha y}{\sqrt{2}} - \left(\frac{P}{2} + E_T\right)\alpha C_1 \frac{\alpha y}{\sqrt{2}} \sin\frac{\alpha y}{\sqrt{2}}\right\} \sin\alpha x \\ \dot{\tau}^{22} &= \left\{\left(\frac{P}{2} + E_T\right)\sqrt{2}\alpha C_1 \frac{\alpha y}{\sqrt{2}} \cos\frac{\alpha y}{\sqrt{2}} + C_2 \sin\frac{\alpha y}{\sqrt{2}}\right\} \cos\alpha x. \end{aligned} \tag{6.12}$$

For the symmetric "bulging" mode, Fig. 2,

$$u = \sqrt{2}C_1 \cos\frac{\alpha y}{\sqrt{2}} \sin \alpha x$$
$$v = -C_1 \sin\frac{\alpha y}{\sqrt{2}} \cos \alpha x \tag{6.13}$$

$$\dot{t}^{11} = \left\{\left[\left(2P+E_T\right)\sqrt{2}\alpha C_1 + \tfrac{1}{2}C_2\right] . \cos\frac{\alpha y}{\sqrt{2}} - \left(\frac{P}{2}+E_T\right)\frac{\alpha}{\sqrt{2}}C_1\frac{\alpha y}{\sqrt{2}}\sin\frac{\alpha y}{\sqrt{2}}\right\} \cos \alpha x$$
$$\dot{t}^{12} = \left\{\left[\left(\frac{3P}{2}+E_T\right)\alpha C_1 + \frac{1}{\sqrt{2}}C_2\right] \sin\frac{\alpha y}{\sqrt{2}} + \left(\frac{P}{2}+E_T\right)\alpha C_1\frac{\alpha y}{\sqrt{2}}\cos\frac{\alpha y}{\sqrt{2}}\right\} \sin \alpha x \tag{6.14}$$
$$\dot{t}^{22} = \left\{-\left(\frac{P}{2}+E_T\right)\sqrt{2}\alpha C_1\frac{\alpha y}{\sqrt{2}}\sin\frac{\alpha y}{\sqrt{2}} + C_2\cos\frac{\alpha y}{\sqrt{2}}\right\} \cos \alpha x.$$

In (6.4), (6.12), (6.13) and (6.14), C_1 and C_2 are arbitrary constants. The boundary conditions (6.8) on $\dot{t}^{12}$ and $\dot{t}^{22}$ can be satisfied for $C_1, C_2 \neq 0$, provided

$$\pm\frac{\sin\sqrt{2}\alpha h}{\sqrt{2}\alpha h} = \frac{1+\frac{1}{2}.(P/E_T)}{1+\frac{3}{2}.(P/E_T)} \tag{6.15}$$

In (6.15), the plus sign on the left hand side applies for the antisymmetric mode, and the minus sign for the symmetric mode. The left hand side is shown in Fig. 3 ($\phi = \sqrt{2}\alpha h = \sqrt{2}\pi h/L$) and the right hand side in Fig. 5 ($S = P/E_T$).

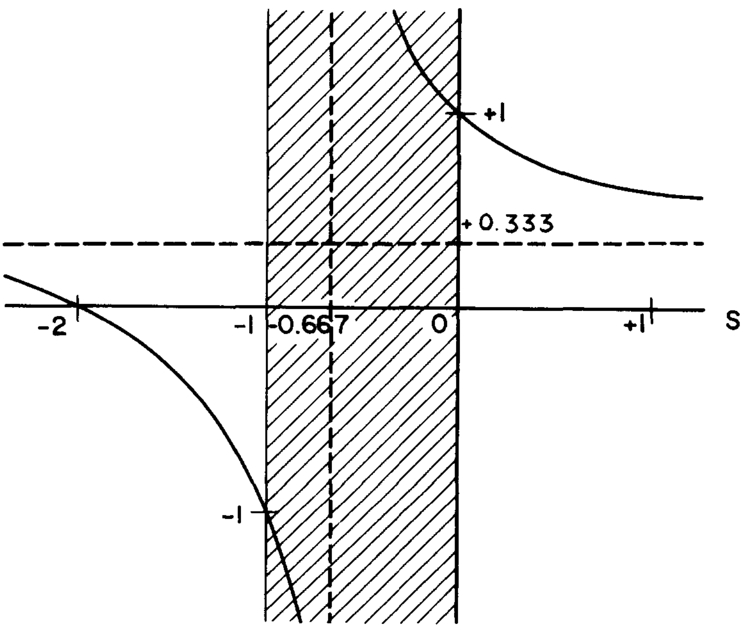

FIG. 5. The function $\frac{1+S/2}{1+3S/2}$.

For $S > 0$ (compression), solutions of (6.15) exist only for the antisymmetric mode. Retaining the leading terms in the power series expansions in h/L and P/E_T leads to the tangent modulus buckling stress for a narrow bar, $h/L \ll 1$,

$$\frac{P}{E_T} = \frac{\pi^2 h^2}{3L^2} \tag{6.16}$$

The eigencondition (6.15) admits solutions for a compressed bar for

$$0 < \frac{P}{E_T} < \infty, \qquad 0 < \frac{h}{L} < 0{\cdot}51,$$

and P/E_T increases from zero to infinity as h/L increases.

For $S < 0$ (tension), the symmetric mode is the mode of instability for least tensile stress σ ($\sigma = -P$). The limiting value of σ for necking as $h/L \to 0$ is given by (6.15) as $\sigma = E_T$, which is the well known result. Hill [9] shows from his uniqueness condition that for $E_T/\sigma < 1$ (with the appropriate change in notation) uniqueness is no longer assured and remarks that stronger inequalities can likely be found for certain boundary conditions and dimensions. The eigencondition (6.15) which applies to thin bars of arbitrary width indicates that σ increases steadily as h/L increases until $h/L = 1{\cdot}01$ (then $\sigma = 4{\cdot}50E_T$), and then σ oscillates slightly as h/L increases further. There are solutions for the antisymmetric mode in tension for

$$1{\cdot}47 < \frac{\sigma}{E_T} < \infty, \qquad 0{\cdot}51 < \frac{h}{L} < \infty.$$

For a thin, flat bar of sufficient length, the critical stress for necking in plane stress ($\sigma = E_T$) is less than the critical stress for necking in plane strain ($\sigma = 4E_T/3$), so such a bar is more susceptible to necking across the width than through the thickness.

A plane stress solution for a rigid–plastic material obeying the von Mises yield criterion has been given by Goodier [6]. The analysis is based on the Kirchhoff stress tensor which coincides with the true stress tensor used here, in virtue of the condition of incompressibility. Hence the boundary conditions and equations of equilibrium in [6] are identical to (6.8) and (6.10). The velocity fields considered are the same as (6.4) and (6.13). The only difference lies in the constitutive relations employed. Equation (6.11) coincides with the constitutive relation in [6] when $P = 0$. This single difference leads to eigenconditions quite different from (6.15). The eigenconditions in [6], (equations (37) and (48)) do not admit solutions for uniform tension in state B; hence necking is excluded. Also, the symmetric "bulging" mode in compression discussed in [6] for plane stress does not occur according to (6.15), although it does for plane strain, as discussed in Section 5.

The principle of objectivity admits other designations of stress rate besides (2.11). The stress rates $\dot{t}^{ij}$ are themselves objective, and when the stress rates $\dot{t}^{ij}$ are used in (2.10) in place of the stress rates $\hat{\sigma}_{ij}$, the constitutive relation for the plastic strain rates used in [6] is obtained. The use of the stress rates $\dot{t}^{ij}$ in (2.10) can be based on some special results of the general theory of Section 5 in [1] which follow from consideration of a rigid–plastic material.

7. DISCUSSION

The stability of inelastic columns has been treated very generally by Hill and Sewell [13–15], with attention being given to shear deformation. However, the subsequent work by Sewell [16, 17] on the instability of inelastic plates neglects shear deformation. The shear stiffening effect in plates, and the symmetric modes which describe necking in tension or bulging in compression have not been explored extensively. The eigencondition (4.14) accounts for these additional aspects of the plate buckling problem.

The boundary conditions in the present treatment have the feature that the tangential component of velocity v over the compressed edges of the plate is not specified. The shear stress obtained in the solution vanishes over these edges, $\theta_1 = x = \pm L$, so there must be no restraint due to friction. The velocity fields considered represent the difference between two possible fields, the deformation corresponding to one being uniform and the other buckled. Since the normal component of velocity u vanishes over the compressed edges, as well as the shear stress, the difference in the rate of doing work for the two fields is zero. The solution represents buckling in the Shanley sense which occurs under sustained flow when there is no reversal in the direction of straining. The amplitudes of the eigenfields can be chosen arbitrarily small, so this condition can always be met. The treatment here corresponds to the "linear" solid of Hill and Sewell. The "non-linear" solid which covers Engesser–Karman buckling in which reversal occurs has not been considered.

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APPENDIX

In order to justify the constitutive relations (2.6) and (2.7), it is convenient to define a symmetric covariant tensor g''_{KL} by

$$g''_{KL} = \delta_{KL} + 2e''_{KL} \tag{A.1}$$

and to denote its symmetric contravariant conjugate by g''^{KL}. If the continuum can be unloaded from state B without reverse plastic flow or residual stresses resulting, then g''_{KL}

is the covariant metric tensor of the unloaded state B_0 referred to the convected coordinate system Y_M. It is now assumed that the elastic constants of the material are not altered by plastic deformations and that the material remains isotropic at all times. The Hooke's law relation for reloading from the hypothetical unloaded state B_0 referred to the convected cordinate system Y_M is (cf. [8], equation (5.4.32))

$$Ee'_{KL} = [(1+\nu)g''_{KM}g''_{LN} - \nu g''_{KL}g''_{MN}]S^{MN}, \tag{A.2}$$

where S^{MN} is the symmetric contravariant Kirchhoff stress tensor. The constitutive relation (A.2) has the form postulated in [1] (equation (5.3)). The inverse of (A.2) is

$$S^{KL} = \frac{E}{1+\nu}\left[g''^{KM}g''^{LN} + \frac{\nu}{1-2\nu}g''^{KL}g''^{MN}\right]e'_{MN}. \tag{A.3}$$

Then from (A.3) it follows that

$$\frac{\partial S^{KL}}{\partial e'_{MN}} = \frac{\partial S^{MN}}{\partial e'_{KL}}. \tag{A.4}$$

Hence S^{KL} can be related to a scalar function A, the Helmholtz function which for isothermal deformations depends on e'_{KL} and e''_{KL}, by

$$S^{KL} = \rho_0 \frac{\partial A}{\partial e'_{KL}} \tag{A.5}$$

where ρ_0 is the initial density, as required by equation (6.11) of [1]. The constitutive relation (A.2) has now been justified on thermodynamic grounds.

The constitutive relations (2.6) and (2.7) are approximations of (A.2) suitable for describing material behaviour when the tangent modulus is small compared to Young's modulus and when volume changes in the plastic part of the deformation are zero. The first condition is the basis of the approximation (2.5) and the additional approximation

$$\dot{g}_{KL} \simeq \dot{g}''_{KL} \tag{A.6}$$

which enters into the derivation of (2.7) from (2.6). According to the second condition, the substitution of the true stress tensor T^{KL} for the Kirchhoff stress tensor S^{KL} in (A.2) is consistent with the approximation (2.5). These approximations greatly simplify subsequent computation.

The term "elastic" has not been used in the strict sense of [1]. Its use here is considered appropriate because the constitutive equation for e'_{KL} incorporates the elastic constants of the material.

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Абстракт—Выводятся определяющие зависимости, в пригодной форме, с целью использования их в задачах устойчивости для упругопластического материала, подвергающегося возмущении из односного напряженного состояния, основанные на обшей теории Грина и Нагди для упругопластического континуума (1), Рассматриваются квазистатическая и изотермическая деформации, материал переходит в пластическое состояние согласно критерия Мизеса и подвергается пластическому течению согласно уравнениям Рейсса. Скорости упругих деформаций определяются по закону Гука. Затем эти зависимости применяются для исследования влияния толщины пластинки на неустойчивость прямоуголрной пластинки, подвергающейся одномерному одноосному сжатию. Рассматривается неустойчивость повержности сжатого полубесконечного упруго-пластического континуума в качестве граничого случая толщины бесконечной пластинки. Констатируется, что пластическое течение в небольшой степени влияет на неустойчивость поверхности, ибо она подвергается упругой деформации. На конец, обсуждаются два случая для жестко-пластического материала, плоской деформации и плоского напряжения, с целью сравнения настоящего исследования с предыдущей работой.